ESTIMATING π AS AN INTRODUCTION TO TECHNICAL COMPUTATION

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Abstract

Engineering technology students often begin their college experience with minimal training in technical computing. A simple exercise in computing an approximate value of π can introduce basic concepts and commonly used software, while calculating a familiar number. Several methods of approximation are presented here, starting with an intuitive geometric approach and progressing to more efficient methods based on series expansions. Initial presentation of this exercise in a math review class for engineering technology students was successful.

Background

Perhaps the most universally used mathematical constant, π appears in expressions across a wide range of technical fields. Every engineering technology student knows that π is approximately equal to 3.14 or 22/7. Given a dozen decimal places, it is possible to express the circumference of a circle 1 million kilometers in diameter with an accuracy of 1mm. While there is no engineering reason to need more than about a dozen decimal places, the numerical value of π is currently known to more than a trillion decimal places. There are several reasons for this incredible level of precision. One is that number theorists are looking for patterns in the series of digits, though π has been proven to be irrational and, thus, cannot terminate in a repeating series of digits no matter how long the number. Another more practical use is to verify the accuracy of new computers by showing that they can correctly calculate π to a large number of digits [1].

Engineering technology students are generally just told that π is the ratio of the circumference of a circle to its diameter and that it has a certain approximate value. Common scientific calculators have π programmed to around a dozen significant digits and some students even take it upon themselves to memorize them. However, is it very uncommon for students to be shown where the numerical value of π comes from and even more rare for them to do the actual calculation.

This situation offers an opportunity to both introduce technology students to some basic ideas about technical calculations and to remove some of the mystery surrounding this universal constant. Fortunately, several means of calculating π are well within the mathematical abilities of technology students who are not yet familiar with the basics of calculus, though a basic knowledge of calculus allows more efficient methods to be used. Many different methods have been proposed over the last few thousand years [2], [3]; what follows is a description of several representative approaches in roughly increasing order of sophistication, along with sample calculations.

Method of Polygons

Among the first known approximations of π are 25/8 (3.125) from the Babylonians and 256/81 (approximately 3.1605) from the Egyptians [4]. Both appear to date from around 1900 BC. The first known rigorous estimate was by Archimedes (287-212 BC). He showed that π can be approximated by inscribing and circumscribing regular polygons on a circle, as shown in Figure 1. As the number of sides increases, the sum of the lengths of the sides approaches the circumference of a circle. The inscribed polygon approaches the circumference of the circle from below, while the circumscribed polygon approaches from above. This method is an important one because it allows calculation to any accuracy desired by simply increasing the number of sides of the polygons.

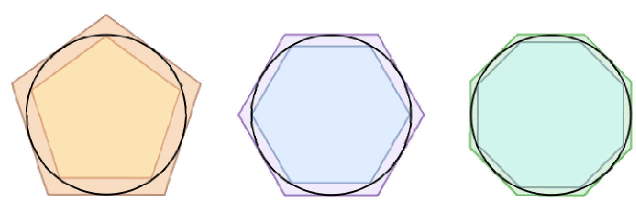


Figure 1. Archimedes' Method of Inscribed and Circumscribed Polygons (Wikipedia Commons. Image is in the public domain)

The circumference of a regular polygon with N sides is N times the length of the base of one segment, as shown in Figure 2. The geometry is slightly different for the inscribed and circumscribed triangles, though both are isosceles. For simplicity, assume the radius of the circle is 1.

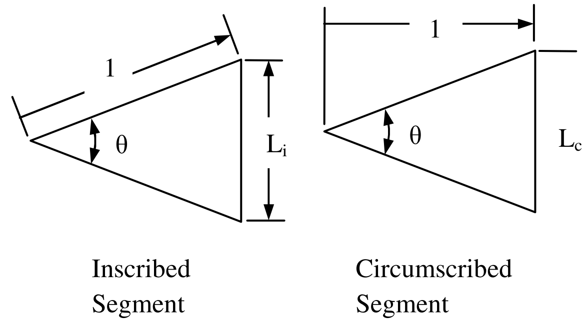


Figure 2. Dimensions of Polygon Segments

For the inscribed polygon with N sides,

$$\sin\left(\frac{\theta}{2}\right)=\frac{L_i}{2} \quad \Rightarrow \quad L_i = 2\sin\left(\frac{\theta}{2}\right) \tag{1}$$

where θ = 360°/N. For the circumscribed polygon,

$$\tan\left(\frac{\theta}{2}\right)=\frac{L_c}{2} \quad \Rightarrow \quad L_c = 2\tan\left(\frac{\theta}{2}\right) \tag{2}$$

If the radius of the circle is 1, then the circumference is 2π and

$$N\sin\left(\frac{\theta}{2}\right)\le \pi \le N\tan\left(\frac{\theta}{2}\right) \tag{3}$$

Of course, calculating an estimate for π depends on the ability to calculate sine and tangent functions. Figure 3 shows how the upper- and lower-bound estimates converge as N increases. The average of the two estimates produces a more accurate result than either estimate individually.

Before moving to the next method, it is worth noting that Equation 3 is true even when N is not an integer. For example, choosing N=123.456 gives the estimate 3.141254 ≤ π ≤ 3.1412271. To go one step further, N does not even have to be real. For example, N = 123.456 + 89.0123i gives 3.141522 + 2.116587i × 10^{-4} ≤ π ≤ 3.141734 + 4.233687i × 10^{-4}. Note that using complex arguments gives complex results whose imaginary parts are very small. As the magnitude of the complex argument increases, the imaginary part of the result approaches zero and the result is a real number. This a useful example of how an equation developed from a simple, intuitive starting point can be applied more generally than its derivation might suggest.

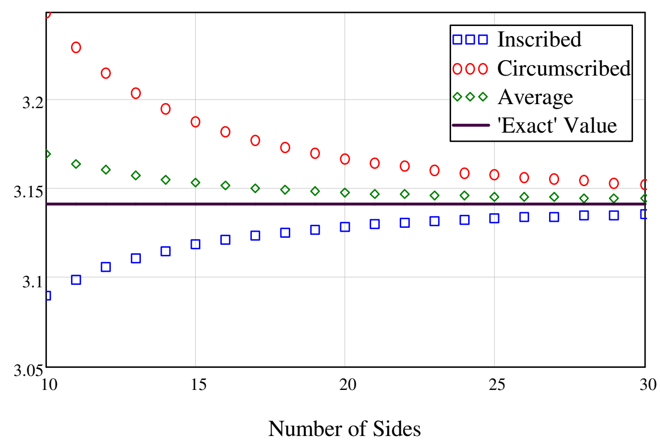


Figure 3. Convergence of Polygonal Approximations

Summation of Boxes

Another approximate method appears to have been proposed slightly before the development of calculus and will look familiar to any student who has seen a graphical explanation of integration [5]. The area of a circle is πr^2 and the equation describing a circle with a radius of 1 is $x^2+y^2=1$. It is easy in principle to approximate the area of a quarter circle by dividing it into vertical boxes (as shown in Figure 4) and simply adding up the areas of the boxes. As the number of boxes increases, the boxes' total area approaches the exact area of the quarter circle. This is essentially the definition of an integral.

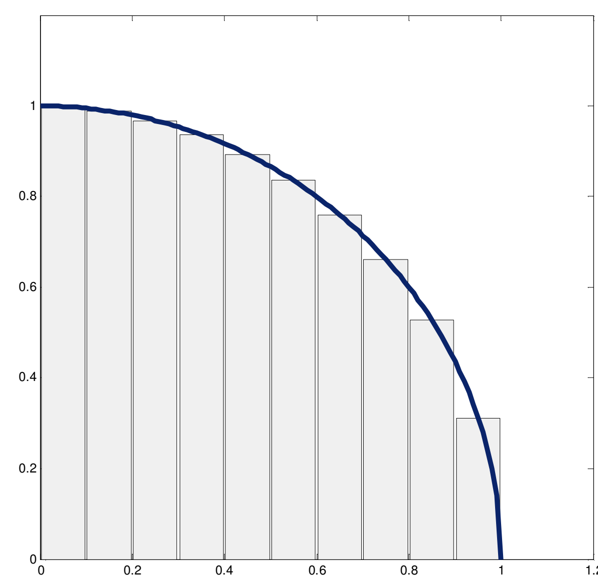


Figure 4. Summing Rectangles to Approximate Area of Quarter Circle

The expression for the area of the quarter circle is

$$A=\sum_{i=1}^{N}\sqrt{1-x_i^2}\,\Delta x=\sum_{i=1}^{N}\sqrt{1-\left[\left(i-\tfrac{1}{2}\right)\frac{1}{N}\right]^2}\,\frac{1}{N} \qquad (4)$$

Since the radius of our circle is 1, π = 4A. Figure 5 shows convergence of this expression compared to Archimedes' method. It clearly produces an accurate result faster and does so without the need to calculate trigonometric functions. However, this comes at the price of having to calculate square roots.

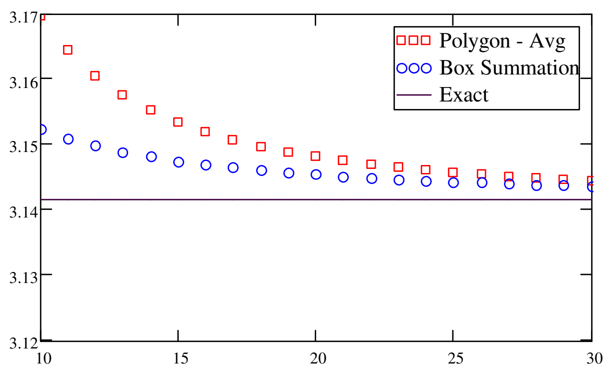


Figure 5. Convergence of Summed Rectangle Estimate

Summation of Segment Lengths

A more sophisticated approach is to sum the length of arc segments on the quarter circle as shown in Figure 6. The quarter circle can be divided into segments with equal spacing along the horizontal axis (Δx is constant). The length of any individual segment is

$$L_i=\sqrt{\Delta x_i^2+\Delta y_i^2} \qquad (5)$$

so the approximate length of the quarter circle is then

$$\frac{C}{4}=\frac{\pi r}{2}=\sum_{i=1}^{N}\sqrt{\Delta x_i^2+\Delta y_i^2} \qquad (6)$$

Since the radius of the circle is 1, Δx=1/N and x_i=i/N. Now, this expression can be re-written as

$$\begin{aligned}\pi&=2\sum_{i=1}^{N}\sqrt{(x_i-x_{i-1})^2+(y_i-y_{i-1})^2}\\&=2\sum_{i=1}^{N}\sqrt{\left(\frac{1}{N}\right)^2+\left(\sqrt{1-\left(\frac{i}{N}\right)^2}-\sqrt{1-\left(\frac{i-1}{N}\right)^2}\right)^2}\end{aligned} \qquad (7)$$

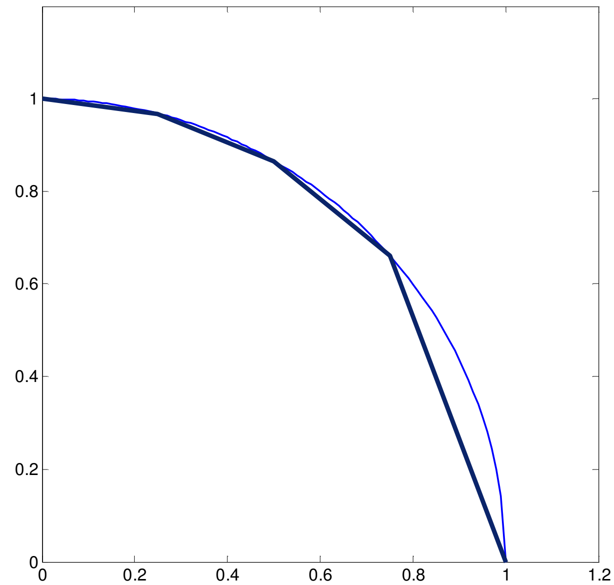


Figure 6. Defining Lengths of Segments

As N gets larger, this expression converges to π, as shown in Figure 7.

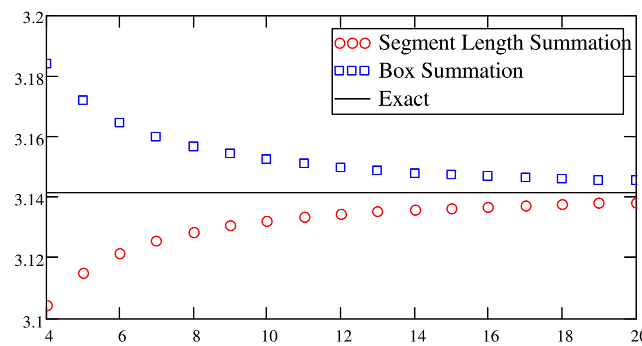
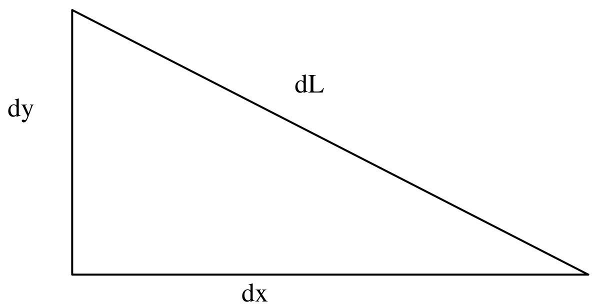


Figure 7. Convergence of Summed Segment Length Method

The summation methods presented in this section and the previous approach are clearly precursors to an integral method. To complete this line of thought, it makes sense to extend the method to a true integral. The length of an infinitesimal section of an arc is simply defined using the Pythagorean Theorem (Figure 8).



$$dL = \sqrt{dx^2 + dy^2}$$
$$= \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx$$

Figure 8. Length of an Infinitesimal Arc Segment

$$\frac{dy}{dx} = \frac{-x}{\sqrt{1-x^2}} \quad and \quad \left(\frac{dy}{dx}\right)^2 = \frac{x^2}{1-x^2}$$
$$so \quad \sqrt{1+\left(\frac{dy}{dx}\right)^2} = \sqrt{\frac{1}{1-x^2}} \tag{8}$$

Since the circumference of the full circle is 2π, the arc length of the quarter circle is π/2 and

$$2\int_0^1 \sqrt{\frac{1}{1-x^2}}\,dx = \pi \tag{9}$$

The methods shown so far require an entirely new calculation for each increase in N. While this may seem like a trivial distinction, it is important when calculating a large number of significant figures. A more useful approach would be one in which increasing the number of terms in the estimate meant simply adding new terms to ones that had already been computed. This leads to infinite series approximations.

Series Expansion Approximations

The purpose of this exercise is to develop an expression that allows calculating approximations to π of arbitrary accuracy. A common method for doing this is a Taylor series expansion or its simpler variation, the Maclaurin series expansion [6]. Many efficient methods of calculating numerical approximations to π are based on these series.

Most beginning calculus students learn that a Taylor series can be used to approximate a continuous function as long as the function is sufficiently smooth. They may even know that scientific calculators use series approximations internally to evaluate familiar functions like sin(x) and cos (x). The general expression for the Taylor series is

$$f(x) = f(a) + \frac{f'(a)}{1!}(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \frac{f'''(a)}{3!}(x-a)^3 + \ldots \tag{10}$$

where "a" is a constant (the number about which the series is expanded). If a = 0, the Taylor series becomes a Maclaurin series.

$$f(x) = f(0) + \frac{f'(0)}{1!}x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \cdots \tag{11}$$

Technology students tend to be more receptive to concepts that can be presented graphically. As an example, the Maclaurin series expansion for tan(x) is

$$\tan(x) \approx x + \frac{1}{3}x^3 + \frac{2}{15}x^5 + \frac{17}{315}x^7 + \cdots \tag{12}$$

Figure 9 shows tan(x) in the range 0.5 ≤ x ≤ 1.5 along with increasingly accurate series approximations. It is clear that increasing the number of terms in the approximation produces approximating functions increasingly close to tan (x).

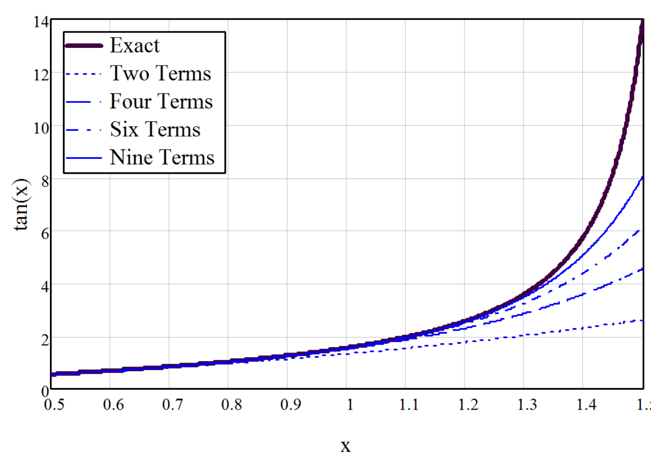


Figure 9. Series Approximations to tan(x)

Being able to approximate a range of expressions with an infinite series opens a new class of functions to be used for approximating π. Since tan(45º) = tan(π/4) = 1, one of the first, and perhaps the most obvious, of these was $\tan^{-1}(1)$ = π/4. Written as a Taylor series, $\tan^{-1}(x)$ is

$$\tan^{-1}(x) = \tan^{-1}(a) + \frac{1}{1+a^2}(x-a) + \frac{-a}{(1+a^2)^2}(x-a)^2 + \left[\frac{8a^2}{(1+a^2)^3} - \frac{2}{(1+a^2)}\right]\frac{(x-a)^3}{3!} + \cdots \quad (13)$$

An approximate expression for $\pi = 4\tan^{-1}(1)$ is, thus,

$$\pi = 4\left[1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} + \cdots + \frac{-1^{N+1}}{2N-1}\right] \quad (14)$$

Figure 10 shows the convergence of this series. It is unique among the series discussed so far in that it alternately overestimates and underestimates the correct value. More important, though, is the fact that the series converges very slowly. This problem is well known and there are other series that converge more quickly.

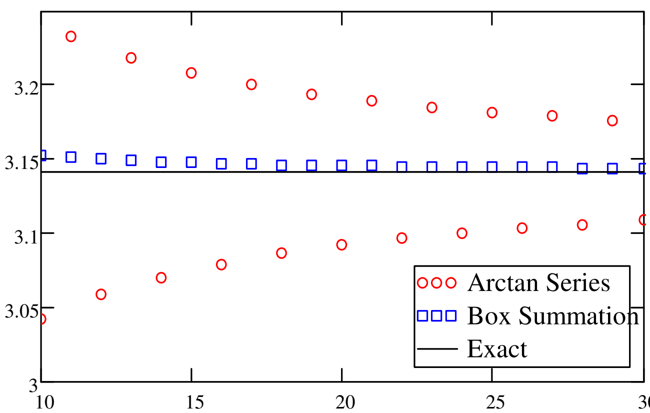


Figure 10. Convergence of Maclaurin Series for Arctan Approximation

Any change to the process that increases the rate at which successive terms decrease in magnitude should increase the rate of convergence. One possibility is to use an angle smaller than 1 radian so that the magnitude of the x^n terms in the Maclaurin series decreases much faster. This may require a change in the function being considered. Specifically, $\sin(\pi/6) = ½$, so $6\sin^{-1}(1/2) = \pi$. The Maclaurin series expression is

$$\pi = 6\sin^{-1}\left(\frac{1}{2}\right) = 6\sum_{n=1}^{\infty}\frac{(2n)!}{4^n(n!)^2(2n+1)}\left(\frac{1}{2}\right)^{2n+1} = 6\left[\frac{1}{2} + \frac{1}{48} + \frac{3}{1280} + \frac{5}{14336} + \frac{35}{589824} + \cdots\right] \quad (15)$$

Note how quickly this series converges, as shown in Figure 11. A six-term series yields 3.1415767, about 99.9995% of the right answer. Since π is irrational, there is no exact numerical value and no way to precisely state the difference between the calculated number and the accepted value. For our purposes, the "exact" answer is assumed to be the value programmed into Mathcad, which is correct to seventeen decimal places.

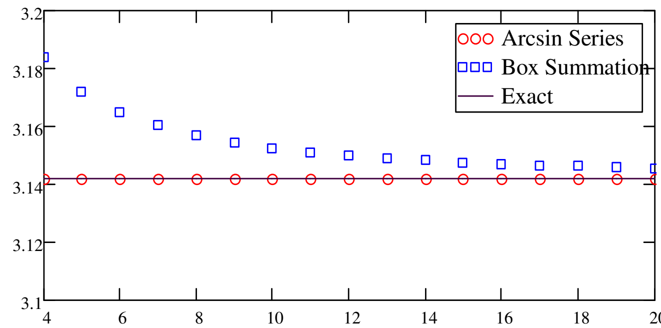


Figure 11. Convergence of Maclaurin Series Expression for 6sin^{-1}(1/2)

Finally, let's revisit the integral expression in Equation 9. The integrand can be expressed using a Maclaurin series approximation.

$$\sqrt{\frac{1}{1+x^2}} = 1 + \frac{x^2}{2!} + \frac{3^2x^4}{4!} + \frac{5^23^2x^6}{6!} + \frac{7^25^23^2x^8}{8!}\cdots \quad (16)$$

There are a number of different integration limits that could be used, but some converge faster than others. Equation 9 calculates the circumference of a quarter circle, so the range of integration is 0 to 1. Since the integral of Equation 16 is a summation of terms of the form cx^n, the upper integration boundary affects the rate of convergence. If the upper limit of integration is 1, then the each term reduces to $c\times1^n = c$. However, if the upper limit is less than 1, then the magnitude decreases much more quickly as the value of the exponent, n, increases.

The constraint is that the upper limit must correspond to some rational, preferably integral, multiple of the circumference of a circle. Fortunately, an upper integration limit of ½ corresponds to an angle of 30° and the resulting arc length is 1/12 the length of the circumference. The resulting expression is

$$\pi = 6\int_0^1 1 + \frac{x^2}{2} + \frac{3^2 x^4}{4!} + \frac{5^2 3^2 x^6}{6!} + \frac{7^2 5^2 3^2 x^8}{8!} + \cdots$$

$$= 6\left[\frac{1}{2} + \frac{1}{3!}\left(\frac{1}{2}\right)^3 + \frac{3^2}{5!}\left(\frac{1}{2}\right)^5 + \frac{5^2 3^2}{7!}\left(\frac{1}{2}\right)^7 + \cdots\right] \quad (17)$$

$$= 6\sum_{n=0}^{N} \frac{\prod_{i=0}^{n}(2i-1)^2}{(2n+1)!}\left(\frac{1}{2}\right)^{2n+1}$$

While this may look like a new expression, a little algebra shows that it is identical to Equation 15. It is interesting that two completely separate arguments can yield the same series approximation, a fact that clearly suggests the two starting expressions are related.

More Advanced Methods

More curious students may want to know about the most advanced methods of calculating π. As this is written, it is now known to approximately 1.24 trillion digits. For such a huge calculation, efficiency is critical. There are a number of different expressions, but most involve series expansions of trigonometric expressions. A particularly efficient expression was proposed by Machin [3].

$$\pi = 16\tan^{-1}\left(\frac{1}{5}\right) - 4\tan^{-1}\left(\frac{1}{239}\right) \quad (18)$$

This is written as a series expression by substituting the MacLaurin series approximation for the arctan function.

$$\tan^{-1} x = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \ldots \quad (19)$$

A four-term series using this expression differs from the "exact" value by less than one part in 10^6.

A more recent expression was developed by Takano [7].

$$\pi = 48\tan^{-1}\left(\frac{1}{49}\right) + 128\tan^{-1}\left(\frac{1}{57}\right) - 20\tan^{-1}\left(\frac{1}{239}\right) + 48\tan^{-1}\left(\frac{1}{110443}\right) \quad (20)$$

A four-term series using this expression differs from the "exact" value by less than one part in 10^{14}.

The origin of "Machin-like" formulae lies in complex numbers. This is a particularly useful feature, since complex numbers seem to be a source of continual confusion among engineering technology students. Students were shown how to derive one of these formulae by starting with the simple expression, (2+i)(3+i) = 5+5i. It is a simple task to present this expression graphically in the real-imaginary plane. Presented graphically in polar form, the two numbers are represented in Figure 12 below.

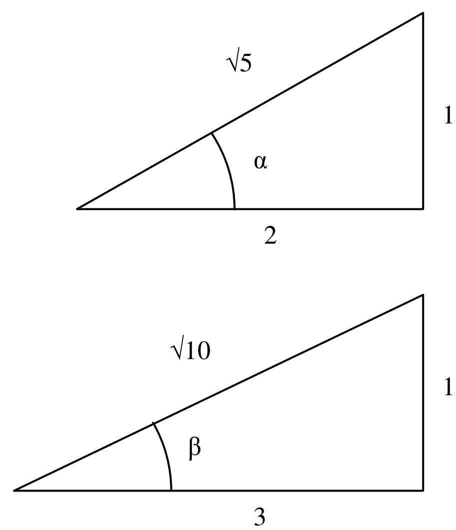


Figure 12. Polar Representation of Complex Numbers

In polar notation, the complex multiplication is

$$\sqrt{5}\angle\alpha \times \sqrt{10}\angle\beta = \sqrt{50}\angle\alpha + \beta \quad (21)$$

5+5i = √50 Đπ/4, so α+β = π/4 and

$$\pi = 4\left[\tan^{-1}\left(\frac{1}{2}\right) + \tan^{-1}\left(\frac{1}{3}\right)\right] \quad (22)$$

This expression is slightly less efficient than the one in Equation 15. However, it leads directly to the other, more efficient expressions such as Equation 18. Figure 13 shows a comparison of convergence between the arcsin expression from Equation 15, Machin's expression from Equations 18 and 22. The vertical axis shows the magnitude of the difference between the N-term series approximation and the value of π programmed into Mathcad.

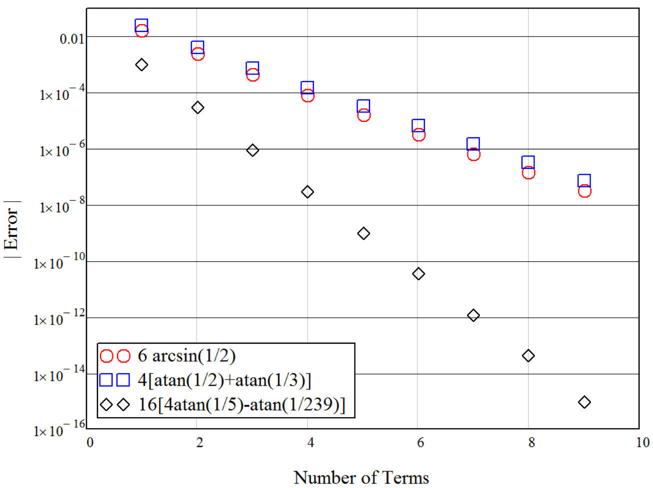


Figure 13. Comparison of Convergence for Inverse Trigonometric Expressions

Calculation Using Familiar Mathematical Tools

In order for the idea of calculating π to be made real to engineering technology students, the students need to do the calculations themselves and see how the individual steps work. While any of the methods presented here could be implemented on a scientific calculator, this is a chance to use software tools. The examples presented here will use Equations 4 and 15.

While not ideally suited to technical calculations, spreadsheet programs such as Excel can be useful, particularly when simple plots are needed. Figure 14 shows a sample calculation using the arctan and arcsin series methods. Because of the great differences in the rates of convergence, it makes sense to plot the magnitude of the individual terms in each series. The terms of the arcsin series decrease much more rapidly than the arctan series. By the 10^{th} term, they are about seven orders of magnitude apart. This is a very compelling way of showing students graphically how differences in the formulation of the problem affect how much effort it takes to calculate a precise result.

Figure 15 shows a typical calculation using Mathcad. The nature of the program allows students to write out the series expressions in a clear format. It is easy to plot the convergence of the box summation method directly since the number of boxes can be specified as a parameter of the function being plotted. Additionally, Mathcad includes symbolic manipulation tools so that students can develop series expressions symbolically if they wish.

Index	Arctan Series Terms	Magnitude of Arctan Terms	Arcsin Series Terms	
0	4.00000	4.00000	3.00000	
1	-1.33333	1.33333	0.12500	
2	0.80000	0.80000	0.01406	
3	-0.57143	0.57143	0.00209	
4	0.44444	0.44444	3.56038E-04	
5	-0.36364	0.36364	6.55434E-05	
6	0.30769	0.30769	1.27095E-05	
7	-0.26667	0.26667	2.55704E-06	
8	0.23529	0.23529	5.28799E-07	
9	-0.21053	0.21053	1.11713E-07	
10	0.19048	0.19048	2.40049E-08	
11	-0.17391	0.17391	5.23033E-09	
12	0.16000	0.16000	1.15285E-09	
13	-0.14815	0.14815	2.56600E-10	
14	0.13793	0.13793	5.75927E-11	
15	-0.12903	0.12903	1.30203E-11	
16	0.12121	0.12121	2.96224E-12	
17	-0.11429	0.11429	6.77706E-13	
18	0.10811	0.10811	1.55816E-13	
19	-0.10256	0.10256	3.59839E-14	
20	0.09756	0.09756	8.34322E-15	
	3.18918		3.141592654	<= Estimates of PI

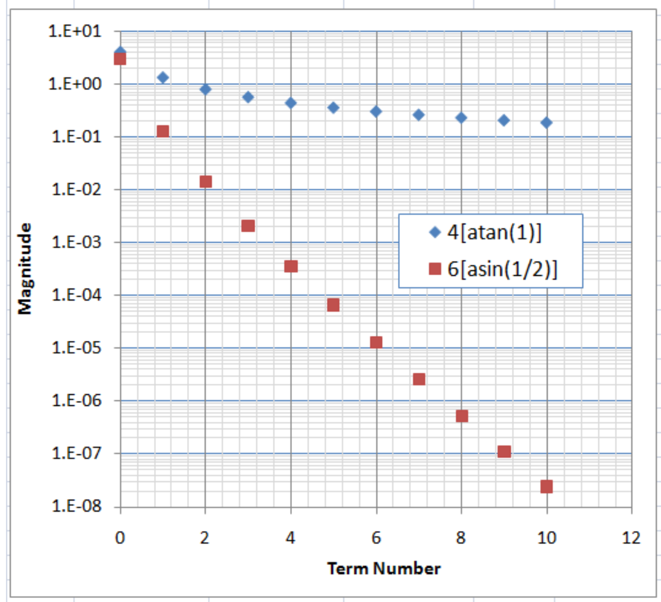


Figure 14. Example Calculation using Excel

Student Response

The calculations presented here were used in a mathematics review class as a learning module. An informal survey of students after the end of the class showed several themes in the students' perception of this exercise. The overall response from the students was positive and they clearly wanted to see this module remain in the syllabus. The first theme in the student responses was the clear utility of the result. While the mathematical concepts underlying the various calculation methods presented here could be easily presented in the context of a made-up problem, the central role of π in technical calculations clearly created interest among the students.

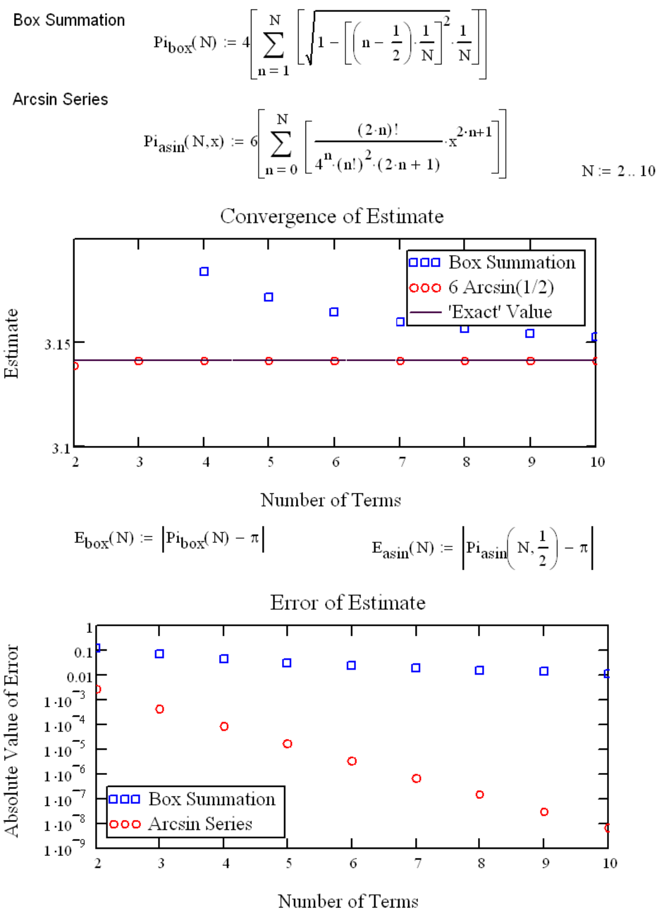


Figure 15. Example Calculation Using Mathcad

A closely related theme in the comments was that students liked knowing what the answer should be when they started making their own calculations. In response, I pointed out that I could have given them the answer beforehand for any problem I had assigned. However, this wasn't enough for them. Their familiarity with the number π appears to be important. They all clearly knew what π was and many had memorized its value to more decimal places than they would ever need, but none of them knew where the numerical value had come from. Being able to solve this little mystery clearly motivated some of the students.

There were two other comments worth mentioning. One was that calculating π presented a clear example in which complex numbers were useful and necessary. The other was that the idea of accelerating convergence was an example of optimization and increasing the efficiency of a process.

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Biography

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